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HOLOMORPHIC VECTOR BUNDLES ON OPEN SUBSETS OF STEIN MANIFOLDS

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ABSTRACT

Let M be a connected two-dimensional Stein manifold with $H^2(M, \mathbf{Z}) = 0$ and $\mathbf{S} \subset \mathbf{M}$ a discrete subset with $S \neq \emptyset$. Set $X := M \setminus S$. Fix an integer $r \geq 2$. Then there exists a rank r vector bundle E on X such that there is no line bundle L on X with a non-zero map $L \to E$.

1. Introduction

By the Oka–Grauert principle for any Stein space W the holomorphic and the topological classification of complex vector bundles on W coincide ([G]). Several papers were devoted to the study of holomorphic vector bundles on complex analytic spaces. Every vector bundle E on a Stein space has holomorphic sections and in particular there is a line bundle A with a non-zero map $A \to E$. The same is true on every projective variety, but may be false on a compact two-dimensional complex manifold ([BL]). In [B2] we considered holomorphic vector bundles on $\mathbf{C}^2 \setminus \{0\}$ and proved in a very different way the case $M = \mathbf{C}^2$, $S = \{0\}$ and $r = 2$ of the following result.

THEOREM 1.1: *Let M be a connected two-dimensional Stein manifold with $H^2(M, \mathbf{Z}) = 0$ and $S \subset M$ a discrete subset with $S \neq \emptyset$. Set $X := M \setminus S$. Fix an integer $r \geq 2$. Then there exists a rank r vector bundle E on X such that there is no line bundle A on X with a non-zero map $A \to E$.*

The same proof will give the following result.

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Theorem 1.2: *Let M be a connected complex manifold with $H^1(X, \mathbf{O}_M) = H^2(M, \mathbf{Z}) = 0$ and $S \subset M$ a closed analytic subset such that each of its irreducible components has codimension at least two in M. Set $X := M \setminus S$ and assume $H^1(X, \mathbf{O}_X) \neq 0$. Fix an integer $r \geq 2$. Then there exists a rank r vector bundle E on X such that there is no line bundle A on X with a non-zero map $A \to E$.*

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2. Proofs of Theorems 1.1 and 1.2

We recall that a finite holomorphic map is a proper holomorphic map whose fibers are finite. We will use only the particular case "Y smooth" of the following result.

Proposition 2.1: *Let X be a smooth connected complex manifold, Y a reduced equidimensional locally Cohen–Macaulay complex space such that every $R \in \mathrm{Pic}(Y)$ with $H^0(Y, R) \neq 0$ is trivial and $f\colon Y \to X$ a surjective finite holomorphic map. Assume that the induced map $f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is not surjective and take $L \in \mathrm{Pic}(Y) \setminus \mathrm{Im}(f^*(\mathrm{Pic}(X)))$. Set $r := \deg(f)$. Then $f_*(L)$ is a rank r vector bundle on X such that for every $A \in \mathrm{Pic}(X)$ we have $H^0(X, \mathrm{Hom}(A, f_*(L))) = 0$.*

Proof: Since Y is locally Cohen–Macaulay, X is smooth and f is finite, f is flat. Thus E is locally free. The rank of E is computed on any non-empty open subset U of X such that $f^{-1}(U)$ is the disjoint union of d open subsets of Y, each of them mapped biholomorphically onto U by f. Assume the existence of $A \in \mathrm{Pic}(X)$ such that there is $h\colon A \to f_*(L)$ with $h \neq 0$. The map h induces a non-zero map $h'\colon f^*(A) \to L$. Thus $H^0(Y, L \otimes f^*(A)^*) \neq \emptyset$. By our assumption on Y the line bundle $L \otimes f^*(A)^*$ is trivial, i.e., $L \cong L \otimes f^*(A)^*$. Thus $L \in \mathrm{Im}(f^*(\mathrm{Pic}(X)))$, a contradiction.

Example 2.2: Fix an integer $r \geq 2$. Let X be a connected complex manifold such that every effective Cartier divisor on X is trivial but X has a non-trivial line bundle, R. Let Y be the disjoint union of r copies of X, say $Y = X_1 \cup \cdots \cup X_r$ with $X_i \cong X$ for every i, and let $f\colon Y \to X$ be the finite holomorphic map which sends each connected component X_i biholomorphically onto X. Let L be the unique line bundle on Y such that $L|X_1 \cong R$ and $L|X_i$ is trivial for every $i \geq 2$. By Proposition 2.1, $f_*(L)$ is a rank r vector bundle on X such that there is no line bundle A on X with $H^0(X, \mathrm{Hom}(A, f_*(L))) \neq 0$.

Proof of Theorem 1.1: By [S2], Th. 2.15, every effective Cartier divisor D on X is the restriction to X of a Cartier divisor Δ on M. Since M is Stein, we have $H^1(X, \mathbf{O}_M) = 0$. Since $H^2(M, \mathbf{Z}) = 0$, the exponential sequence for the Picard group of M implies that every line bundle on M is trivial. Thus $\mathbf{O}_M(\Delta) \cong \mathbf{O}_M$. Thus $\mathbf{O}_X(D) \cong \mathbf{O}_X(\Delta)|M \cong \mathbf{O}_M$. Since X is an open subset of a two-dimensional Stein manifold, we have $H^2(X, \mathbf{O}_X) = 0$ ([S1]). Since X is not Stein but an open subset of a Stein manifold, we have $H^1(X, \mathbf{O}_X) \neq 0$ (see, e.g., [B1] or [Co]). Hence the exponential sequence of X implies the existence of a non-trivial line bundle on X. Hence we may apply Example 2.2.

Proof of Theorem 1.2: We just assumed that M, S and X have all the properties used in the proof of the Theorem.

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